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TRADE-OFFS BETWEEN PARTS OF THE OBJECTIVE FUNCTION OF A LINEAR PROGRAM

TECHNICAL REPORT NO. ESD-TR-65-363

DECEMBER 1965

H. C. Joksch

Prepared for

DIRECTORATE OF COMPUTERS
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

Project 7070

Prepared by

THE MITRE CORPORATION
Bedford, Massachusetts
Contract AF 19(628)-2390

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ABSTRACT

Linear programs whose objective function can be separated into two parts are considered. The following problem is studied: given an optimal solution with respect to the total objective function, how does it have to be modified to change one part of the objective function by a certain amount and affect the other part as little as possible?

The construction of an optimal simplex-tableau for the modified problem from the optimal simplex-tableau of the original problem is demonstrated. This can serve as a starting tableau for an ordinary parametric programming procedure. Finally, the marginal exchange ratio for both parts of the objective function and the simplex-multipliers for the constraints are derived.

REVIEW AND APPROVAL

This technical report has been reviewed and is approved.

John B. Curtis
JOHN B. CURTIS
1st Lt, USAF
Project Officer

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SECTION I

INTRODUCTION

In some linear programming models, the objective function is composed of parts which actually represent subobjectives. A solution which is optimal with respect to the sum of the parts might no longer be optimal if the relative weights attached to the different subobjectives are changed. Of course, one could consider this as a new problem and solve it independently from the original one. However, it might be desirable to find the new solution for an entire range of modified weights. Also, if the changes are small, it is probable that the optimal solution of the new problem does not differ too much from that of the old problem. Therefore, the following question is studied: is there a simple way to modify an optimal solution to change one part of the objective function by a certain amount so that the remaining part of the objective function is least affected?

SECTION II

FORMULATION OF THE PROBLEM

Let the original problem (I) be to maximize

$$\underline{c}^1\underline{x}^1 + \underline{c}^2\underline{x}^2 = z \tag{1}$$

under the constraints

$$A^1\underline{x}^1 + A^2\underline{x}^2 = \underline{b} \tag{2}$$

$$\underline{x}^i \geq 0 \ , \tag{3}$$

and let $\hat{\underline{x}}^1$ and $\hat{\underline{x}}^2$ be an optimal solution. The modified problem (II) is: What is the optimal solution of

$$\underline{c}^2\underline{x}^2 = z^2 = \max \tag{4}$$

under the constraints

$$\underline{c}^1\underline{x}^1 = \hat{z}^1 + \delta \ , \tag{5}$$

where $\hat{z}^1 = \underline{c}^1\hat{\underline{x}}^1$,

$$A^1\underline{x}^1 + A^2\underline{x}^2 = \underline{b} \tag{6}$$

and the non-negativity condition (3), for small values of δ?

SECTION III

CONSTRUCTION OF AN OPTIMAL SIMPLEX-TABLEAU FOR THE MODIFIED PROBLEM

Obviously, for $\delta = 0$, $\hat{\underline{x}}^1$ and $\hat{\underline{x}}^2$ are also optimal solutions of Problem II. Therefore, one can start with the optimal simplex-tableau of Problem I and obtain from it one for Problem II for $\delta = 0$. Tableau (7) represents the main body of the optimal simplex-tableau for I; $(\hat{y}^1_{ij})$ and $(\hat{y}^2_{ij})$ are the parts originating from A^1 and A^2, respectively.

$$\begin{array}{|c|c|c|}\hline \hat{y}_{io} & \hat{y}^1_{ij} & \hat{y}^2_{ij} \\ \hline \end{array} \qquad \begin{array}{|c|}\hline \hat{\beta}_{ij} \\ \hline \end{array} \; . \qquad (7)$$

The matrix (β_{ij}) is the basic inverse belonging to this tableau. A tableau for Problem II has to contain an additional row for constraint (5); it originates from row (8)

$$\begin{array}{|c|c|c|c|}\hline z^2 & \delta & c^1_j & 0 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline \end{array} \qquad (8)$$

through the sequence of transformations which generated (7). Instead of going through these transformations the same result can be achieved by subtracting appropriate multiples of the rows of (7) from (8), so that the elements of (8) in the basis columns become zero. If B^1 is the set of the x^1_i, B^2 that of the x^2_i in the optimal basis, and $\sum^k$ indicates summation over the rows belonging to the x^k_i of B^k, the transformation will yield

$$
\begin{array}{|c|c|c|c|}\hline 0 & \delta & c_j^1 - \sum^{l} y_{ij}^1 c_i^1 & -\sum^{l} y_{ij}^2 c_i^1 \\ \hline \end{array}
\quad
\begin{array}{|c|c|}\hline -\sum^{l} \hat{\beta}_{ij} c_i^1 & 1 \\ \hline \end{array}
\qquad (9)
$$

which shall be abbreviated by

$$
\begin{array}{|c|c|c|c|}\hline 0 & \delta & h_j^1 & h_j^2 \\ \hline \end{array}
\quad
\begin{array}{|c|c|}\hline h_j^3 & 1 \\ \hline \end{array}
\ . \qquad (10)
$$

By definition, the h_j^k for the basic variables are zero. Since Problem II has one more constraint than I, one more variable has to be introduced into the basis. This variable will be zero for $\delta = 0$. Assume that the pivot will be h_t^1; then (10) will be transformed into

$$
\begin{array}{|c|c|c|c|}\hline 0 & \delta/h_t^1 & h_j^1/h_t^1 & h_j^2/h_t^1 \\ \hline \end{array}
\quad
\begin{array}{|c|c|}\hline h_j^3/h_t^1 & 1/h_t^1 \\ \hline \end{array}
\qquad (11)
$$

and (7) will be transformed into

$$
\begin{array}{|c|c|c|c|}\hline \hat{y}_{io} & -\delta\dfrac{\hat{y}_{it}}{h_t^1} & \hat{y}_{ij}^1 - \dfrac{h_j^1}{h_t^1}\hat{y}_{it}^1 & \hat{y}_{ij}^2 - \dfrac{h_j^2}{h_t^1}\hat{y}_{it}^1 \\ \hline \end{array}
\quad
\begin{array}{|c|c|}\hline \hat{\beta}_{ij} - \dfrac{h_j^3}{h_t^1}\hat{y}_{it}^1 & -\dfrac{\hat{y}_{it}^1}{h_t^1} \\ \hline \end{array}
\ . \qquad (12)
$$

Thus, (11) and (12) are the main body of the optimal simplex-tableau belonging to Problem II.

The objective row for this tableau will be obtained by transformation of the row

$$
\begin{array}{|c|c|c|c|}\hline 0 & 0 & 0 & c_j^2 \\ \hline \end{array}
\quad
\begin{array}{|c|c|}\hline 0 & 0 \\ \hline \end{array}
\qquad (13)
$$

in a manner similar to that of obtaining (9) from (8), resulting in

$$\begin{array}{|c|c|c|c|}\hline -\hat{z}^2 & 0 & k_j^1 & k_j^2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline k_j^3 & 0 \\ \hline \end{array} \tag{14}$$

where

$$k_j^1 = - \sum^2 \hat{y}_{ij}^1 c_i^2 \tag{15}$$

$$k_j^2 = c_j^2 - \sum^2 \hat{y}_{ij}^2 c_i^2 \tag{16}$$

$$k_j^3 = - \sum^2 \hat{\beta}_{ij} c_i^2 \tag{17}$$

We note that

$$h_j^1 + k_j^1 = \hat{y}_{oj}^1 \leq 0 \tag{18}$$

$$h_j^2 + k_j^2 = \hat{y}_{oj}^2 \leq 0 \tag{19}$$

and

$$h_j^3 + k_j^3 = - \hat{\pi}_j \quad , \tag{20}$$

where $\hat{\pi}_j$ are the simplex multipliers belonging to the optimal solution of I. Tableau (14) is not yet an objective row for the simplex-tableau consisting of (11) and (12), except in the case of $k_t^1 = 0$. Otherwise, it would have to be transformed into zero by subtraction of an appropriate multiple of row (11), which results in

$$\begin{array}{|c|c|c|c|}\hline -\hat{z}^2 & -\delta\dfrac{k_t^1}{h_t^1} & k_j^1 - k_t^1\dfrac{h_j^1}{h_t^1} & k_j^2 - k_t^1\dfrac{h_j^2}{h_t^1} \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline k_j^3 - k_t^1\dfrac{h_j^3}{h_t^1} & -\dfrac{k_t^1}{h_t^1} \\ \hline \end{array} \;\cdot \tag{21}$$

Since it is known that $\underline{\hat{x}}^1$, $\underline{\hat{x}}^2$ is an optimal solution for $\delta = 0$, of Problem II, the conditions

$$k_j^1 - k_t^1 \frac{h_j^1}{h_t^2} \leq 0 \tag{22}$$

$$k_j^2 - k_t^1 \frac{h_j^2}{h_t^1} \leq 0 \tag{23}$$

have to be satisfied.

If the pivot had been one of the h_t^2, (11*), (12*) and (21*) would be the optimal simplex-tableau for Problem II.

$\hat{y}_{io}$	$-\delta \frac{\hat{y}_{it}^2}{h_t^2}$	$\hat{y}_{ij}^1 - \frac{h_j^1}{h_t^2} \hat{y}_{it}^2$	$\hat{y}_{ij}^2 - \frac{h_j^2}{h_t^2} \hat{y}_{it}^2$	$\hat{\beta}_{ij} - \frac{h_j^3}{h_t^2} \hat{y}_{it}^2$	$-\frac{\hat{y}_{it}^1}{h_t^2}$	(12*)
0	δ/h_t^2	h_j^1/h_t^2	h_j^2/h_t^2	h_j^3/h_t^2	$1/h_t^1$	(11*)
$-\hat{z}^2$	$-\delta \frac{k_t^2}{h_t^2}$	$k_j^1 - k_t^2 \frac{h_j^2}{h_t^2}$	$k_j^2 - k_t^2 \frac{h_j^2}{h_t^2}$	$k_j^3 - k_t^2 \frac{h_j^3}{h_t^2}$	$-\frac{k_t^2}{h_t^2}$.	(21*)

The conditions for optimality of the solution are:

$$k_j^1 - k_t^2 \frac{h_j^2}{h_t^2} \leq 0 \tag{22*}$$

$$k_j^2 - k_t^2 \frac{h_j^2}{h_t^2} \leq 0 \quad . \tag{23*}$$

SECTION IV

SELECTION OF A NEW BASIC VARIABLE

From (22), (23), (22*) and (23*) we obtain a criterion for the selection of an h_t^ℓ for pivot:

$$\max_{h_j^1>0} \frac{k_j^1}{h_j^1} \leq \frac{k_t^1}{h_t^1} \leq \min_{h_j^1<0} \frac{k_j^1}{h_j^1} \tag{24}$$

$$\max_{h_j^2>0} \frac{k_j^2}{h_j^2} \leq \frac{k_t^1}{h_t^1} \leq \min_{h_j^2<0} \frac{k_j^2}{h_j^2} \tag{25}$$

for a left pivot.

$$\max_{h_j^1>0} \frac{k_j^1}{h_j^1} \leq \frac{k_t^2}{h_t^2} \leq \min_{h_j^1<0} \frac{k_j^1}{h_j^1} \tag{26}$$

$$\max_{h_j^2>0} \frac{k_j^2}{h_j^2} \leq \frac{k_t^2}{h_t^2} \leq \min_{h_j^2<0} \frac{k_j^2}{h_j^2} \tag{27}$$

for a right pivot. These conditions are satisified because the solution is optimal for $\delta = 0$; part of them, depending upon the sign of the desired change of δ, determines the pivot h_t^ℓ. If $\delta \geq 0$, then $h_t^\ell > 0$ and for

$$\max_{h_j^2>0} \frac{k_j^2}{h_j^2} \leq \max_{h_j^1>0} \frac{k_j^1}{k_j^2} = \frac{k_t^1}{h_t^1} \tag{28}$$

x_t^1 is the new basic variable, for

$$\max_{h_j^1>0} \frac{k_j^1}{h_j^1} \leq \max_{h_j^2>0} \frac{k_j^2}{h_j^2} = \frac{k_t^2}{h_t^2} \tag{29}$$

x_t^2 is the new basic variable. Similarly, for $\delta \leq 0$, $h_t^\ell<0$ and either

$$\frac{k_t^2}{h_t^2} = \min_{h_j^2<0} \frac{k_j^2}{h_t^2} \leq \min_{h_j^1<0} \frac{k_j^1}{h_j^1} \tag{30}$$

or

$$\frac{k_t^1}{h_t^1} = \min_{h_j^1<0} \frac{k_j^1}{h_k^1} \leq \min_{h_j^2<0} \frac{k_j^2}{h_j^2} \tag{31}$$

determines the pivot.

SECTION V

PARAMETRIZATION OF THE SOLUTION

Once the pivot and an optimal tableau for Problem II, are known, the whole sequence of optimal solutions for the interesting range of δ can be determined by the usual parametric programming techniques.

If one is interested only in the modifications caused by small changes in δ, it is sufficient to determine the pivot column and to read the new values of the variables, from (12) and (11) or (12*) and (11*); namely

$$y_{io} = \hat{y}_{io} - \delta\frac{\hat{y}_{it}}{h_t} , \tag{32}$$

and the newly introduced variable

$$y_{to} = - \delta\frac{1}{h_t} . \tag{33}$$

For the objective function we obtain from (21) or (21*)

$$z^2 = \hat{z}^2 + \delta\frac{k_t}{h_t} . \tag{34}$$

Therefore, the marginal rate of substitution between $z^1 = \hat{z}^1 + \delta$ and z^2 is given by the characteristic quotient k_t/h_t which determines the new basic variable.

Finally, the new simplex-multipliers are of interest; (21) or (21*), respectively, gives them as

$$-\pi_j = k_j^3 - k_t \frac{h_j^3}{h_t} \tag{35}$$

$$-\pi_0 = -\frac{k_t}{h_t} \quad . \tag{36}$$

Condition (35) can be explicitly written as

$$-\pi_j = -\sum^1 \hat{\beta}_{ij} c_i^1 + \frac{k_t^1}{h_t^1} \sum^2 \hat{\beta}_{ij} c_i^1 \tag{37}$$

or

$$-\pi_j = -\sum^2 \hat{\beta}_{ij} c_i^2 + \frac{k_t^2}{h_t^2} \sum^1 \hat{\beta}_{ij} c_i^1 \ , \tag{38}$$

respectively, dependent upon the choice of the pivot. This can be written in a more compact form. If the basis-inverse B^{-1} is split so that

$$\hat{\underline{x}}^1 = (B^{-1})_1 \underline{b} \tag{39}$$

and

$$\hat{\underline{x}}^1 = (B^{-1})_2 \underline{b} \ , \tag{40}$$

components $\hat{\underline{\pi}}^1$ and $\hat{\underline{\pi}}^2$ of the simplex-multipliers can be defined by

$$\hat{\underline{\pi}}^1 = (B^{-1})_1^T \underline{c} \tag{41}$$

and

$$\hat{\underline{\pi}}^2 = (B^{-1})_2^T \underline{c} \ . \tag{42}$$

Then (37) and (38) become

$$\pi_j = \hat{\pi}_j^1 - \frac{k_t^1}{h_t^1} \hat{\pi}_j^2 \tag{43}$$

or

$$\pi_j = \hat{\pi}_j^2 - \frac{k_t^2}{h_t^2} \hat{\pi}_j^1 \, . \tag{44}$$

SECTION VI

SUMMARY

To obtain an optimal tableau for the modified problem (as a starting point for a parametric programming procedure), one has to start with an optimal tableau (7) for the original problem, generate the additional rows (10) and (14), and, dependent upon the sign of the desired change δ of the new objective function, apply the criteria (28) and (29), or (30) and (31). This gives the new basic variable (in this tableau still with the value 0). Knowing the new basic variable, one obtains as the new optimal tableau either (12), (11) and (21), or (12*), (11*) and (21*). From here on, standard parametric programming techniques can be applied.

If one does not want to know the complete modified solution, but only how far a change of one part of the objective function influences the other, then it is not necessary to determine the new simplex-tableau. It is sufficient to determine the quantities necessary to evaluate (34).

Similarly, one can obtain the simplex-multipliers for the optimal solution of the modified problem by following the procedure up to the solution of the new basic variable. If one "splits" the expression for the original simplex-multipliers in a manner representing the contributions of both parts of the objective function, then (43) or (44) respectively allows an easy calculation of the new simplex-multipliers for the components of the old ones and the rate of substitution between the objective functions.

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